

Hardy Type Inequalities for Fractional Integrals and Derivatives of Riemann–Liouville

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Abstract—We prove new Hardy type inequalities for Riemann–Liouville fractional integrals and derivatives in the case when the weight function have power and logarithmic singularities.

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1. INTRODUCTION

Hardy type inequalities have many applications in mathematical physics [1–7]. For instance, S.L. Sobolev used Hardy’s inequalities in the theory of embedding functional spaces [6]. In the recent papers [1, 7] F.G. Avkhadiev used them to estimate the torsional rigidity. The results of A. Laptev and T. Veidle from [5], and the results of A. Balinskiy, A. Laptev and A.V. Sobolev from [4] can be applied in the study of the negativity of spectrum of two-dimensional Schrödinger operators. Hardy type inequalities have been extended in different ways (see [1–28]). For example, some authors obtained Hardy’s inequalities for arbitrary weights [8–17]. We note the result of V.D. Stepanov from [15]. For weight functions u and v V. D. Stepanov shows that the following inequality holds:

$$\left(\int_0^\infty |I_{0+}^\alpha f(x)u(x)|^p dx \right)^{1/p} \leq C \left(\int_0^\infty |f(x)v(x)|^p dx \right)^{1/p},$$

where $\alpha > 1$, $p = 2$ and $I_{0+}^\alpha f$ is a Riemann–Liouville operator of fractional integration and C is some constant.

The classical Hardy inequality [18] with a power weight reads as follows

$$\int_0^\infty \frac{F^p}{x^s} dx \leq \left(\frac{p}{|s-1|} \right)^p \int_0^\infty \frac{f^p}{x^{s-p}} dx, \quad p > 1, s \neq 1, \quad (1)$$

where f is absolutely continuous on $[0, \infty)$, and

$$F(x) = \begin{cases} \int_x^x f(t)dt, & \text{if } s > 1, \\ \int_0^\infty f(t)dt, & \text{if } s < 1. \end{cases} \quad (2)$$

Following to Yu.A. Dubinskii [3, 25] the inequality (1) is called a direct Hardy inequality because in (2) the integration limits are from 0 to x when $s > 1$, and the inequality (1) is called a reverse Hardy inequality because in (2) the integration limits are from x to ∞ when $s < 1$.

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